NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 Year 12 Course Assessment Task 4 (Trial Examination)
Monday 30 August, 2021

General instructions

- Working time – 3 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MXX.1 – Mr Sekaran ◯ 12MXX.2 – Ms Ham

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	$\overline{10}$	$\overline{17}$	$\overline{16}$	$\overline{18}$	$\overline{12}$	$\overline{14}$	$\overline{13}$	$\overline{100}$

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 13).

Questions **Marks**

1. Consider the following statement. **1**

$$x > 3 \text{ or } x < -3$$

Which of the following is the negation of the statement above?

(A) $x < 3$ or $x > -3$

(B) $x < 3$ and $x > -3$

(C) $x \leq 3$ or $x \geq -3$

(D) $x \leq 3$ and $x \geq -3$

2. Which of the following is a sixth root of i? **1**

(A) $e^{i\frac{\pi}{6}}$ (B) $e^{i\frac{\pi}{4}}$ (C) $e^{i\frac{3\pi}{4}}$ (D) $e^{i\frac{\pi}{12}}$

3. For a certain complex number z where $\arg(z) = \dfrac{\pi}{5}$, which of the following does $\arg(z^7)$ equal to? **1**

(A) $-\dfrac{7\pi}{5}$ (B) $-\dfrac{3\pi}{5}$ (C) $\dfrac{2\pi}{5}$ (D) $\dfrac{3\pi}{5}$

4. A particle of mass m is moving in a straight line with the following force acting on it: **1**

$$F = \frac{m}{x^3}(6 - 10x)$$

Which of the following is an expression for its velocity in any position x, if the particle starts from rest at $x = 1$?

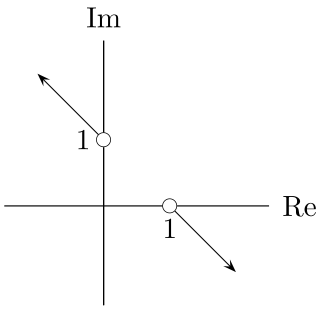
(A) $v = \pm\dfrac{1}{x}\sqrt{-3 + 10x - 7x^2}$

(B) $v = \pm\sqrt{2}x\sqrt{-3 + 10x - 7x^2}$

(C) $v = \pm\dfrac{\sqrt{2}}{x}\sqrt{-3 + 10x - 7x^2}$

(D) $v = \pm\dfrac{\sqrt{2}}{x}\sqrt{-3 + 10x + 7x^2}$

5. The path traced out by a complex number z is shown on the Argand diagram below. **1**



Which of the following is the equation of the path traced by z?

(A) $\arg(z-i)-\arg(z-1)=0$

(B) $\arg(z-i)-\arg(z-1)=\pi$

(C) $\arg(z+i)-\arg(z+1)=0$

(D) $\arg(z-i)-\arg(z-1)=-\pi$

6. Using the substitution $x=\pi-y$, which of the following will the definite integral **1**

$$\int_0^\pi x\sin x\,dx$$

simplify to?

(A) 0

(B) $\displaystyle\int_0^\pi \sin x\,dx$

(C) $\displaystyle\frac{\pi}{2}\int_0^\pi \sin x\,dx$

(D) $\dfrac{\pi^2}{4}$

7. The vector $\underset{\sim}{v}$ is given by $\underset{\sim}{v}=\dfrac{1}{2}\begin{pmatrix}\sqrt{2}\\-1\\1\end{pmatrix}$. Which of the following is the correct description of $\underset{\sim}{v}$? **1**

(A) $\underset{\sim}{v}$ makes an angle of 135° with the positive $x-$axis and 150° with the positive $y-$axis.

(B) $\underset{\sim}{v}$ makes an angle of 45° with the positive $x-$axis and 150° with the positive $y-$axis.

(C) $\underset{\sim}{v}$ makes an angle of 45° with the positive $x-$axis and 120° with the positive $y-$axis.

(D) $\underset{\sim}{v}$ makes an angle of 120° with the positive $y-$axis and 30° with the positive $z-$axis.

8. Given that $x, y \in \mathbb{Z}$, where $x, y \geq 0$, which of the following is a **FALSE** statement? **1**

(A) $\forall x\,(\exists y : y = x)$

(B) $\exists x\,(\exists y : y = 2 - x)$

(C) $\forall x\,(\exists y : y = 1 + 2x)$

(D) $\exists x\,(\forall y : y = 1 - 2x)$

9. A particle is undergoing simple harmonic motion about a fixed point O. At time t seconds it has displacement x metres from O given by $x = a\cos nt$ for some constants $a > 0$ and $n > 0$. The period of the motion is T seconds. **1**

What is the time taken by the particle to move from its starting position to a point half-way towards O?

(A) $\dfrac{T}{12}$ (B) $\dfrac{T}{9}$ (C) $\dfrac{T}{8}$ (D) $\dfrac{T}{6}$

10. Which of these inequalities is **FALSE**? (Do NOT attempt to evaluate the integrals) **1**

(A) $\displaystyle\int_1^2 \frac{1}{1+x}\,dx < \int_1^2 \frac{1}{x}\,dx$

(B) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x}\,dx$

(C) $\displaystyle\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 Marks) Commence a NEW booklet. **Marks**

(a) i. Prove that $\forall p \in \mathbb{Z}^+$, **2**

If p^3 is even then p is even.

ii. Prove that $\sqrt[3]{2}$ is irrational. **3**

(b) Suppose that n and $n+1$ are positive integers, neither of which is divisible by 3. **3**

Prove that $n^3 + (n+1)^3$ is divisible by 9.

(c) The sequence x_n is given by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{4 + x_n}{1 + x_n} \text{ for } n \in \mathbb{Z}^+ \text{ where } n \geq 1$$

i. Prove by mathematical induction that for $n \in \mathbb{Z}^+$ where $n \geq 1$, **4**

$$x_n = 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right)$$

where $\alpha = -\dfrac{1}{3}$

ii. Hence find the limiting value of x_n as $n \to \infty$. **1**

(d) It is given that $a, b \in \mathbb{R}^+$.

i. If $a + b = 6$, show **2**

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{3}$$

ii. If $a + b = c$, show **2**

$$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{c^2}$$

Examination continues overleaf...

Question 12 (16 Marks) Commence a NEW booklet. **Marks**

(a) The points $A(1,-2,3)$ and $B(-5,4,-1)$ lie on the line ℓ_1.

i. Show that a vector equation of ℓ_1 is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. **1**

ii. Consider a line ℓ_2 with parametric equations **3**

$$\begin{cases} x = 1 - \mu \\ y = 2 + 3\mu \\ z = -1 + \mu \end{cases} \qquad \text{where } \mu \in \mathbb{R}$$

Assuming ℓ_2 is neither parallel nor perpendicular to ℓ_1, determine whether ℓ_1 and ℓ_2 intersect or are skew.

(b) A sphere S_1 with centre $C(-3,-5,10)$ passes through the point with coordinates $A(3,-3,6)$.

i. Show that the vector equation of S_1 is **1**

$$\left| \underset{\sim}{u} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right| = 2\sqrt{14}$$

ii. Write down the Cartesian equation of S_1. **1**

iii. The vector equation of another sphere S_2 is **2**

$$\left| \underset{\sim}{r} - \begin{pmatrix} -9 \\ 4 \\ 7 \end{pmatrix} \right| = \sqrt{14}$$

Prove that the two spheres S_1 and S_2 touch each other at a single point.

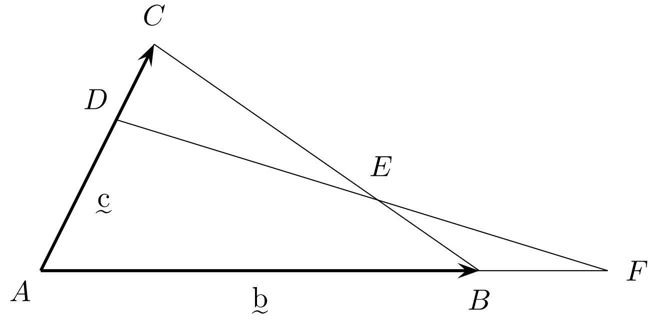
iv. The vector equation of the line m is given as **3**

$$\underset{\sim}{v} = \begin{pmatrix} -6 \\ -3 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \text{where } \lambda \in \mathbb{R}$$

Find the value(s) of λ if the line m intersects the sphere S_1 twice.

Examination continues overleaf...

(c) In $\triangle ABC$ below, D is the point on AC such that $AD : DC = 2 : 1$. E is the point on BC such that $BE : EC = 1 : 2$.



When DE is extended, it meets the extension of AB at F. Let $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AC} = \underset{\sim}{c}$.

i. Show that $\overrightarrow{DE} = \dfrac{2}{3}\underset{\sim}{b} - \dfrac{1}{3}\underset{\sim}{c}$. **2**

ii. Show that $AF : BF = 4 : 1$ **3**

[*Hint*: You may assume that $\overrightarrow{DF}$ is a scalar multiple of $\overrightarrow{DE}$, and $\overrightarrow{AF}$ is a scalar multiple of $\overrightarrow{AB}$]

Examination continues overleaf...

Question 13 (18 Marks) Commence a NEW booklet. **Marks**

(a) It is given that $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$, where p and q are real. **3**

Find the value of p and q.

(b) i. Using De Moivre's theorem, or otherwise, show that for every positive integer n, **3**

$$(1+i)^n + (1-i)^n = \left(\sqrt{2}\right)^{n+2} \cos \frac{n\pi}{4}$$

ii. Hence, or otherwise, show that for every positive integer n divisible by 4, **3**

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots + \binom{n}{n} = (-1)^{\frac{n}{4}} \left(\sqrt{2}\right)^n$$

<u>*Note*</u>: $\binom{n}{r} = {}^nC_k$

(c) Let $\mu = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$. It is given that the complex number $\alpha = \mu + \mu^2 + \mu^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where $a, b \in \mathbb{R}$.

i. Prove that $1 + \mu + \mu^2 + \ldots + \mu^6 = 0$ **1**

ii. The second root of the quadratic equation is β. Show with full working that **2**

$$\beta = \mu^3 + \mu^5 + \mu^6$$

iii. Find the values of the coefficients a and b. **3**

iv. Deduce that **3**

$$-\sin \frac{\pi}{7} + \sin \frac{2\pi}{7} + \sin \frac{3\pi}{7} = \frac{\sqrt{7}}{2}$$

Examination continues overleaf...

Question 14 (12 Marks) Commence a NEW booklet. **Marks**

(a) Find **3**

$$\int \frac{4x+3}{(x^2+1)(x+2)}\, dx$$

(b) i. Given $n \in \mathbb{Z}^+$, show that **1**

$$\sec^{2n}\theta = \sum_{k=0}^{n} \binom{n}{k} \tan^{2k}\theta$$

ii. Hence find **2**

$$\int \sec^8\theta \, d\theta$$

[*Hint:* Write $\sec^8\theta$ as $\sec^6\theta \sec^2\theta$]

(c) i. By using a suitable substitution, or otherwise, evaluate **4**

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\, dx$$

ii. Hence using integration by parts, or otherwise, evaluate **2**

$$\int_0^{\frac{\sqrt{3}}{2}} 3x^2 \cos^{-1} x \, dx$$

Examination continues overleaf…

Question 15 (14 Marks) Commence a NEW booklet. **Marks**

(a) i. Find **2**

$$\int \ln(1+x)\,dx$$

ii. Let $I_n = \displaystyle\int_0^1 x^n \ln(1+x)\,dx$ where $n = 0, 1, 2 \ldots$. **3**

Show that

$$(n+1)I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$$

where $n = 1, 2 \ldots$.

iii. Hence show that **4**

$$(n+1)I_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n+1}$$

when n is odd.

(b) A particle is moving in simple harmonic motion with period T about a centre O. Its displacement at any time t is given by $x = a\sin(nt)$, where a is the amplitude.

i. Show that the velocity of the particle is **1**

$$\dot{x} = \frac{2a\pi}{T}\cos\left(\frac{2\pi}{T}t\right)$$

ii. The point P lies D units on the positive side of O. Let V be the velocity of the particle when it first passes through P. **2**

Show that the first time the particle is at P after passing through O is

$$\frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

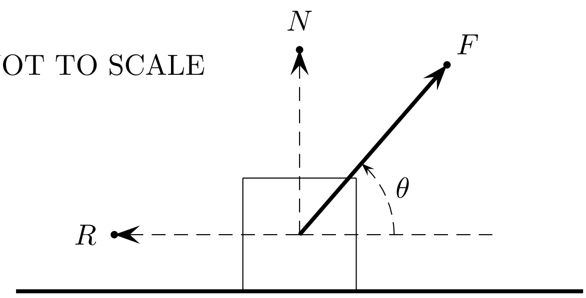
iii. If the second time the particle is at P after passing through O is $t = t_2$, show that **2**

$$\tan\left(\frac{2\pi}{T}t_2\right) = -\frac{2\pi D}{VT}$$

Examination continues overleaf...

Question 16 (13 Marks) Commence a NEW booklet. **Marks**

(a) A block of mass 5 kg is to be moved along a rough horizontal surface by a force of magnitude F newtons, inclined at an angle of θ to the direction of motion, where $0 \leq \theta \leq \dfrac{\pi}{2}$.



There is a frictional force of magnitude R newtons, which is proportional to the normal reaction force of magnitude N newtons exerted on the block by the surface, such that $R = 0.2N$. Take $g = 10\,\mathrm{ms}^{-2}$.

i. Show that when the block is about to move, **3**

$$F = \frac{50}{5\cos\theta + \sin\theta} \text{ newtons}$$

ii. Calculate the minimum value of F needed to overcome the frictional resistance between the block and the surface. **3**

(b) A projectile is launched from the origin with a velocity vector $\dot{\underset{\sim}{r}} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$. It is subject to gravity and there is air resistance, which acts in the opposite direction to the instantaneous direction of motion. The magnitude of the air resistance is mkv, where v is the velocity of the projectile at any time t, and m is the mass of the projectile.

i. Show that the position vector of the projectile is given by **5**

$$\underset{\sim}{r} = \begin{pmatrix} \dfrac{u_0}{k}\left(1 - e^{-kt}\right) \\ \left(\dfrac{g}{k^2} + \dfrac{v_0}{k}\right)\left(1 - e^{-kt}\right) - \dfrac{g}{k}t \end{pmatrix}$$

ii. Hence, or otherwise, find the Cartesian equation of the path of the projectile. **2**

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: ....

Class (please ✔)

◯ 12MXX.1 – Mr Sekaran ◯ 12MXX.2 – Ms Ham

Directions for multiple choice answers

- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.

Ⓐ Ⓑ ● Ⓓ

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

Ⓐ Ⓑ ● ●

- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:

Ⓐ Ⓑ ● ● correct

1 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**6** –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
2 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**7** –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
3 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**8** –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**9** –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
5 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**10** –	Ⓐ	Ⓑ	Ⓒ	Ⓓ

Sample Band E4 Responses

Section I

1. (D) **2.** (C) or (D) **3.** (B) **4.** (C) **5.** (A)
6. (C) **7.** (C) **8.** (D) **9.** (D) **10.** (D)

Section II

Question 11 ()

(a) i. (2 marks)

✓ [1] for $p^3 = (2m+1)^3$.

✓ [1] for final conclusion with reasons

Assume p is odd. Then $\exists m \in \mathbb{Z}^+$ such that $p = 2m+1$.

$$\begin{aligned} p^3 &= (2m+1)^3 \\ &= 8m^3 + 12m^2 + 6m + 1 \\ &= 2(4m^3 + 6m^2 + 3m) + 1 \\ &= 2N + 1, \quad \text{where } N = 4m^3 + 6m^2 + 3m \end{aligned}$$

such that $N \in \mathbb{Z}^+$. Hence p^3 is odd, which is a contradiction.
$\therefore$ If p^3 is even then p is even.

ii. (3 marks)

✓ [1] for cubing to get $2 = \dfrac{p^3}{q^3}$

✓ [1] for deducing q is even

✓ [1] for final conclusion with reasons

Assume $\sqrt[3]{2}$ is rational.

$$\begin{aligned} \sqrt[3]{2} &= \frac{p}{q}, \quad \text{where } p \text{ and } q \text{ are co-prime} \\ 2 &= \frac{p^3}{q^3} \\ 2q^3 &= p^3 \end{aligned}$$

If p^3 is even, then p is even (proven in (i)).
Then $\exists k \in \mathbb{Z}$ such that $p = 2k$.

$$\begin{aligned} 2q^3 &= (2k)^3 \\ 2q^3 &= 8k^3 \\ q^3 &= 4k^3 \end{aligned}$$

$\therefore q^3$ is even, then q is even.
Since both p and q are even they are not co-prime, which is a contradiction.
$\therefore \sqrt[3]{2}$ is irrational.

(b) (3 marks)

✓ [1] for explaining why $n = 3k+1$

✓ [1] for getting $n^3 + (n+1)^3$

✓ [1] for final conclusion with reasons

Since neither n nor $n+1$ is divisible by 3, $\exists k \in \mathbb{Z}^+$ such that $n = 3k+1$ or $n = 3k+2$.
$\therefore$ If $n = 3k+2$ then $n+1 = 3k+3$, which is divisible by 3.
Thus n and $n+1$ must be of the form $3k+1$ and $3k+2$ respectively.

$$\begin{aligned} n^3 + (n+1)^3 &= [n + (n+1)](n^2 - n(n+1) + (n+1)^2) \\ &= (2n+1)(n^2+n+1) \\ &= (6k+3)(6k^2+12k+4+3k+2+1) \\ &= (6k+3)(6k^2+15k+9) \\ &= 9(2k+1)(2k^2+5k+3) \end{aligned}$$

which is divisible by 9, since $2k+1, 2k^2+5k+3 \in \mathbb{Z}$.

(c) i. (4 marks)

✓ [1] for proving the base case.

✓ [2] for the inductive hypothesis, and for progress in using recurrence relation.

✓ [1] for final conclusion.

Let $P(n)$ be the proposition

- Base case: $P(1)$:

$$\begin{array}{ll} \text{LHS} & x_1 = 1 \\ \text{RHS} & 2\left(\dfrac{1+\left(-\dfrac{1}{3}\right)}{1-\left(-\dfrac{1}{3}\right)}\right) = 2\left(\dfrac{\frac{2}{3}}{\frac{4}{3}}\right) = 1 \end{array}$$

  Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true, $k \in \mathbb{Z}^+$:

$$x_k = 2\left(\frac{1+\alpha^k}{1-\alpha^k}\right) \quad \forall k \in \mathbb{Z}^+$$

Prove $P(k+1)$ is true:

$$\begin{aligned}
x_{k+1} &= \frac{4+x_k}{1+x_k} \\
&= \frac{4+2\left(\dfrac{1+\alpha^k}{1-\alpha^k}\right)}{1+2\left(\dfrac{1+\alpha^k}{1-\alpha^k}\right)} \quad \text{...from the assumption} \\
&= \frac{4-4\alpha^k+2+2\alpha^k}{1-\alpha^k+2+2\alpha^k} \\
&= \frac{6-2\alpha^k}{3+\alpha^k} \\
&= \frac{6\left(1-\dfrac{\alpha^k}{3}\right)}{3\left(1+\dfrac{\alpha^k}{3}\right)} \\
&= \frac{2\left(1+\left(-\dfrac{1}{3}\right)\alpha^k\right)}{\left(1-\left(-\dfrac{1}{3}\right)\alpha^k\right)} \\
&= 2\left(\frac{1+\alpha^{k+1}}{1-\alpha^{k+1}}\right)
\end{aligned}$$

$\therefore P(k+1)$ is true.
By Mathematical induction, $P(n)$ is true for $n \in \mathbb{Z}^+$ where $n \geq 1$.

ii. (1 mark)
As $\alpha^n \to 0$ as $n \to \infty$, $\quad x_n \to 2$.

(d) i. (2 marks)
✓ [1] for $ab \leq 9$
✓ [1] for showing the final result
If $a+b=6$,

$$\begin{aligned}
\because a^2+b^2 &\geq 2ab \\
(a+b)^2 &\geq 4ab \\
6^2 &\geq 4ab \\
ab &\leq 9 \\
\frac{1}{ab} &\geq \frac{1}{9} \\
\therefore \frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab} &\geq \frac{6}{9} = \frac{2}{3}
\end{aligned}$$

ii. (2 marks)
✓ [1] for $\dfrac{1}{ab} \geq \dfrac{4}{c^2}$
✓ [1] for showing the final result

If $a+b=c$,

$$\begin{aligned}
\because (a+b)^2 &\geq 4ab \\
c^2 &\geq 4ab \\
\frac{1}{ab} &\geq \frac{4}{c^2} \\
\therefore \frac{1}{a^2}+\frac{1}{b^2} &= \frac{a^2+b^2}{(ab)^2} \\
&\geq \frac{2ab}{(ab)^2} \\
&= \frac{2}{ab} \\
&\geq 2\left(\frac{4}{c^2}\right) \\
&= \frac{8}{c^2}
\end{aligned}$$

Question 12 ()

(a) i. (1 mark)

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix}-5\\4\\-1\end{pmatrix} - \begin{pmatrix}1\\-2\\3\end{pmatrix}\\ &= \begin{pmatrix}-6\\6\\-4\end{pmatrix}\\ &= 2\begin{pmatrix}-3\\3\\-2\end{pmatrix}\end{aligned}$$

$\therefore$ A vector equation is $\underset{\sim}{r} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\2\end{pmatrix}$, where $\lambda \in \mathbb{R}$.

ii. (3 marks)

✓ [1] for setting up three pairs of equations.

✓ [1] for values of λ and μ

✓ [1] for showing the lines are skew by substitution

$\underset{\sim}{r_1} = \begin{pmatrix}1-3\lambda\\-2+3\lambda\\3-2\lambda\end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix}1-\mu\\2+3\mu\\-1+\mu\end{pmatrix}$

$$\begin{cases}1-3\lambda = 1-\mu & ...(1)\\ -2+3\lambda = 2+3\mu & ...(2)\\ 3-2\lambda = 2+3\mu & ...(3)\end{cases}$$

From (1), $\mu = 3\lambda$.
Sub this into (2),

$$\begin{aligned}-2+3\lambda &= 2+9\lambda\\ 6\lambda &= -4\\ \lambda &= -\frac{2}{3} \text{ and } \mu = -2\end{aligned}$$

Sub these into (3),

$$\begin{aligned}\text{LHS } &= 3-2\left(-\frac{2}{3}\right) = \frac{13}{3}\\ \text{RHS } &= 2+3(-2) = -4\end{aligned}$$

$\therefore$ LHS $\neq$ RHS, the lines are skew.

(b) i. (1 mark)

radius $r = \sqrt{6^2+2^2+4^2} = \sqrt{56} = 2\sqrt{14}$

$\therefore$ The vector equation is $\left|\underset{\sim}{u} - \begin{pmatrix}-3\\-5\\10\end{pmatrix}\right| = 2\sqrt{14}$

ii. (1 mark)
$(x+3)^2 + (x+5)^2 + (x-10)^2 = 56$

iii. (2 marks)
✓ [1] for finding the distance between the two centres
✓ [1] for showing the final result
Distance between the two centres $(-3,-5,10)$ and $(-9,4,7)$ is

$$\sqrt{6^2+9^2+3^2} = \sqrt{126} = 3\sqrt{14} = \sqrt{14} + 2\sqrt{14}$$

, which is the addition of two radii.
$\therefore S_1$ and S_2 touch each other at a single point.

iv. (3 marks)
✓ [1] for equating the two vector equations
✓ [1] for the quadratic equation
✓ [1] for two values of λ

$$\begin{aligned}
\text{Equate } \underset{\sim}{v} = \begin{pmatrix} -6+2\lambda \\ -3+\lambda \\ 11+\lambda \end{pmatrix} \text{ and } \left| \underset{\sim}{u} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right| &= 2\sqrt{14} \\
\left| \begin{pmatrix} -6+2\lambda \\ -3+\lambda \\ 11+\lambda \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right| &= 2\sqrt{14} \\
\left| \begin{pmatrix} -3+2\lambda \\ 2+\lambda \\ 1+\lambda \end{pmatrix} \right| &= 2\sqrt{14} \\
(-3+2\lambda)^2 + (2+\lambda)^2 + (1+\lambda)^2 &= 2\sqrt{14} \\
9 - 12\lambda + 4\lambda^2 + 4 + 4\lambda + \lambda^2 + 1 + 2\lambda + \lambda^2 &= 56 \\
6\lambda^2 - 6\lambda - 42 &= 0 \\
\lambda^2 - \lambda - 7 &= 0 \\
\therefore \lambda &= \frac{1 \pm \sqrt{29}}{2}
\end{aligned}$$

(c) i. (2 marks)
✓ [1] for $\overrightarrow{CE}$
✓ [1] for showing the final result

$$\begin{aligned}
\overrightarrow{DC} &= \frac{1}{3}\underset{\sim}{c} \text{ and } \overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c} \\
\overrightarrow{CE} &= \frac{2}{3}(\underset{\sim}{b} - \underset{\sim}{c}) \\
\therefore \overrightarrow{DE} &= \overrightarrow{DC} + \overrightarrow{CE} \\
&= \frac{1}{3}\underset{\sim}{c} + \frac{2}{3}(\underset{\sim}{b} - \underset{\sim}{c}) \\
&= \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}
\end{aligned}$$

ii. (3 marks)

✓ [1] for getting another expression of $\overrightarrow{AF}$

✓ [1] for values of μ and λ

✓ [1] for showing the final result

Let $\overrightarrow{AF} = \lambda\, \overrightarrow{AB} = \lambda\, \underset{\sim}{b}$ and $\overrightarrow{DF} = \mu\, \overrightarrow{DE} = \dfrac{\mu}{3}(2\underset{\sim}{b} - \underset{\sim}{c})$ where $\lambda, \mu \in \mathbb{R}$.
Also

$$\overrightarrow{AF} = \overrightarrow{AD} + \overrightarrow{DF} = \frac{2}{3}\,\underset{\sim}{c} + \frac{\mu}{3}(2\underset{\sim}{b} - \underset{\sim}{c}) = \frac{2-\mu}{3}\,\underset{\sim}{c} + \frac{2\mu}{3}\,\underset{\sim}{b}$$
$$\therefore \frac{2-\mu}{3} = 0 \text{ and } \frac{2\mu}{3} = \lambda$$
$$\mu = 2 \text{ and } \lambda = \frac{4}{3}$$

Sub $\lambda = \dfrac{4}{3}$ into $\overrightarrow{AF} = \lambda\, \overrightarrow{AB}$

$$3\overrightarrow{AF} = 4\overrightarrow{AB}$$
$$\therefore 3AF = 4AB = 4(AF - BF)$$
$$AF = 4BF$$
$$\therefore AF : BF = 4 : 1$$

Question 13 (Ham)

(a) (3 marks)

✓ [1] for finding the third root

✓ [1] for value of p.

✓ [1] for value of q.

As $P(x)$ has real coefficients, then any complex roots that appear also have its conjugate appear as a root. Hence, $1+i$ and $1-i$ are roots of $z^3 + pz^2 + qz + 6 = 0$. Let the third root be α.

Product of roots:

$$(1+i)(1-i)\alpha = -6$$
$$2\alpha = -6$$
$$\therefore \alpha = -3$$

Sum of roots,

$$(1+i) + (1-i) + (-3) = -p$$
$$\therefore p = 1$$

Sum of product of two roots,

$$(1+i)(1-i) - 3(1+i) - 3(1-i) = q$$
$$2 - 6 = q$$
$$\therefore q = -4$$

Hence roots are $1 \pm i$ and $1 \pm 2i$.

(b) i. (3 marks)

- ✓ [1] for correctly converting $1+i$ and $1-i$ to $e^{i\theta}$ form
- ✓ [1] simplifying $e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}$ to $2\cos\dfrac{n\pi}{4}$
- ✓ [1] for showing the final result

$$\begin{aligned}(1+i)^n + (1-i)^n &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n + \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n \\ &= \left(\sqrt{2}\right)^n \left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right) \\ &= \left(\sqrt{2}\right)^n \left(2\cos\frac{n\pi}{4}\right) \\ &= \left(\sqrt{2}\right)^n \times \left(\sqrt{2}\right)^2 \cos\frac{n\pi}{4} \\ &= \left(\sqrt{2}\right)^{n+2} \cos\frac{n\pi}{4}\end{aligned}$$

ii. (3 marks)

- ✓ [1] for expanding using binomial theorem.
- ✓ [1] for simplifying and equating with the expression in (i).
- ✓ [1] for showing the final result

$$\begin{aligned}(1+i)^n + (1-i)^n &= 1 + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \dots + \binom{n}{n}i^n + \\ &\quad 1 - \binom{n}{1}i + \binom{n}{2}i^2 - \binom{n}{3}i^3 + \dots + \binom{n}{n}i^n \\ &= 2\left[\binom{n}{0} + \binom{n}{2}i^2 + \binom{n}{4}i^4 + \dots + \binom{n}{n}i^n\right] \\ &= 2\left[\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \dots + \binom{n}{n}\right] \\ \therefore 2\left[\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \dots + \binom{n}{n}\right] &= \left(\sqrt{2}\right)^{n+2}\cos\frac{n\pi}{4} \qquad \text{... using (i)} \\ &= \left(\sqrt{2}\right)^{n+2}\cos(m\pi), \qquad \text{where } n = 4m \\ \therefore \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \dots + \binom{n}{n} &= \left(\sqrt{2}\right)^n (-1)^m \\ &= \left(\sqrt{2}\right)^n (-1)^{\frac{n}{4}}\end{aligned}$$

(c) i. (1 mark)

$$\mu^7 = \cos 2\pi + i\sin 2\pi = 1$$
$$\mu^7 - 1 = 0$$
$$(\mu - 1)(1 + \mu + \mu^2 + \dots + \mu^6) = 0$$

Since

$$\mu \neq 1, \qquad 1 + \mu + \mu^2 + \dots\mu^6 = 0$$

ii. (2 marks)

✓ [1] for $\beta = \overline{\mu} + \overline{\mu}^2 + \overline{\mu}^4$
✓ [1] for showing the final result.

Since α is a complex root, $\bar{\alpha}$ is also a root.

$$\beta = \overline{\alpha} = \overline{\mu + \mu^2 + \mu^4} = \overline{\mu} + \overline{\mu}^2 + \overline{\mu}^4$$

Since

$$\begin{aligned}\overline{\mu} &= \cos\left(-\frac{2\pi}{7}\right) + \sin\left(-\frac{2\pi}{7}\right) = \cos\frac{12\pi}{7} + \sin\frac{12\pi}{7} = \mu^6 \\ \overline{\mu}^2 &= \cos\left(-\frac{4\pi}{7}\right) + \sin\left(-\frac{4\pi}{7}\right) = \cos\frac{10\pi}{7} + \sin\frac{10\pi}{7} = \mu^5 \\ \overline{\mu}^4 &= \cos\left(-\frac{8\pi}{7}\right) + \sin\left(-\frac{8\pi}{7}\right) = \cos\frac{6\pi}{7} + \sin\frac{6\pi}{7} = \mu^3 \\ &\therefore \beta = \mu^3 + \mu^5 + \mu^6\end{aligned}$$

iii. (3 marks)
✓ [1] for value of a
✓ [1] for expanding and simplifying $\alpha\beta$
✓ [1] for value of b

Using sum of roots,

$$\begin{aligned}a &= -(\alpha + \beta) \\ &= -(\mu + \mu^2 + \mu^3 + \dots + \mu^6) \\ &= -(-1) \\ &= 1\end{aligned}$$

Using product of roots,

$$\begin{aligned}b &= \alpha\beta \\ &= (\mu + \mu^2 + \mu^4)(\mu^3 + \mu^5 + \mu^6) \\ &= \mu^4 + \mu^6 + \mu^7 + \mu^5 + \mu^7 + \mu^8 + \mu^7 + \mu^9 + \mu^{10} \\ &= \mu^4 + \mu^6 + 1 + \mu^5 + 1 + \mu + 1 + \mu^2 + \mu^3 \\ &= 3 + \mu + \mu^2 + \mu^3 + \dots + \mu^6 \\ &= 2\end{aligned}$$

iv. (3 marks)
✓ [1] for solving the quadratic equation
✓ [1] for evaluating α
✓ [1] for showing the final result

The quadratic equation is now $x^2 + x + 2 = 0$.

$$\begin{aligned}x &= \frac{-1 \pm \sqrt{1-8}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2}\end{aligned}$$

Consider $\alpha = \mu + \mu^2 + \mu^4$,

$$\begin{aligned}
\text{Im}(\mu + \mu^2 + \mu^4) &= \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} \\
&= 1.3228... \\
&> 0 \\
\therefore \alpha &= \frac{-1+\sqrt{7}i}{2}
\end{aligned}$$

Now equate the imaginary parts

$$\begin{aligned}
\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + + \sin\frac{8\pi}{7} &= \sin\frac{2\pi}{7} + \sin\left(\pi - \frac{3\pi}{7}\right) + \sin\left(\pi + \frac{\pi}{7}\right) \\
&= \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} \\
&= \frac{\sqrt{7}}{2}
\end{aligned}$$

Question 14 ()

(a) (3 marks)

- ✓ [1] for values of A, B and C
- ✓ [1] for finding the primitive of $\int \frac{x+2}{x^2+1}\,dx$
- ✓ [1] for finding the primitive of $\int \frac{1}{x+2}\,dx$

$$\text{Let } \frac{4x+3}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}, \qquad \text{where } A, B, C \in \mathbb{R}$$
$$4x+3 = (Ax+B)(x+2) + C(x^2+1)$$

Let $x = -2$

$$-5 = 5C,\ C = -1$$

Let $x = 0$

$$3 = 2B + C,\ B = 2$$

Compare the coefficients of x^2

$$0 = A + C,\ A = 1$$

$$\begin{aligned}
\therefore \int \frac{4x+3}{(x^2+1)(x+2)}\,dx &= \int \frac{x+2}{x^2+1} - \frac{1}{x+2}\,dx \\
&= \int \frac{x}{x^2+1} + \frac{2}{x^2+1} - \frac{1}{x+2}\,dx \\
&= \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x - \ln|x+2| + C
\end{aligned}$$

(b) i. (1 mark)

$$\begin{aligned}\sec^{2n}\theta &= \left(1+\tan^2\theta\right)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\tan^2\theta\right)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta\end{aligned}$$

ii. (2 marks)

✓ [1] for using (i) to change the integrand.

✓ [1] for final answer.

$$\begin{aligned}\int \sec^8\theta\,d\theta &= \int \sec^6\theta\sec^2\theta\,d\theta \\ &= \int\left(\sum_{k=0}^{3}\binom{3}{k}\tan^{2k}\theta\right)\sec^2\theta\,d\theta \\ &= \sum_{k=0}^{3}\binom{3}{k}\int\tan^{2k}\theta\sec^2\theta\,d\theta \\ &= \sum_{k=0}^{3}\binom{3}{k}\frac{1}{2k+1}\tan^{2k+1}\theta + C\end{aligned}$$

(c) i. (4 marks)

✓ [1] for transforming the differential.

✓ [1] for transforming both limits.

✓ [1] for transforming integrand to an integrable form.

✓ [1] for final answer.

$$\begin{aligned}x &= \sin\theta \\ \therefore dx &= \cos\theta\,d\theta\end{aligned}$$

Transforming the limits,

$$\begin{aligned}x &= \frac{\sqrt{3}}{2} \quad & \theta &= \frac{\pi}{3} \\ x &= 0 & \theta &= 0\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{3}} (1-\cos^2\theta)\sin\theta\,d\theta \\
&= \int_0^{\frac{\pi}{3}} \sin\theta - \sin\theta\cos^2\theta\,d\theta \\
&= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{3}} \\
&= -\frac{1}{2} + \frac{1}{24} - \left(-1 + \frac{1}{3}\right) \\
&= \frac{5}{24}
\end{aligned}$$

ii. (2 marks)

✓ [1] for correctly integrating

✓ [1] for final answer

$$I_n = \int_0^{\frac{\sqrt{3}}{2}} 3x^2\cos^{-1}x\,dx$$

$$\left|\begin{aligned} u &= \cos^{-1}x & v &= x^3 \\ du &= \frac{-1}{\sqrt{1-x^2}} & dv &= 3x^2 \end{aligned}\right.$$

$$\begin{aligned}
I_n &= \left[x^3\cos^{-1}x\right]_0^{\frac{\sqrt{3}}{2}} + \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx \\
&= \left(\frac{3\sqrt{3}}{8}\times\frac{\pi}{6} - 0\right) + \frac{5}{24} \qquad \text{...from (i)} \\
&= \frac{\sqrt{3}\pi}{16} + \frac{5}{24}
\end{aligned}$$

Question 15 ()

(a) i. (2 marks)

✓ [1] for correct application of integration by parts

✓ [1] for final answer.

$$\int \ln(1+x)\,dx$$

$$\left|\begin{aligned} u &= \ln(1+x) & v &= x \\ du &= \frac{1}{1+x} & dv &= 1 \end{aligned}\right.$$

$$\begin{aligned}
\int \ln(1+x)\,dx &= x\ln(1+x) - \int \frac{x}{1+x}\,dx \\
&= x\ln(1+x) - \int 1 - \frac{1}{1+x}\,dx \\
&= x\ln(1+x) - x + \ln(1+x) + C
\end{aligned}$$

ii. (3 marks)

- ✓ [1] for correct application of integration by parts
- ✓ [1] for arriving at $(*)$
- ✓ [1] for showing the final result

$$I_n = \int x^n \ln(1+x)\,dx$$

$$\left|\begin{array}{ll} u = x^n & v = (x+1)\ln(1+x) - x \\ du = nx^{n-1} & dv = \ln(1+x) \end{array}\right.$$

$$\begin{aligned}
I_n &= \left[x^n\Big(x\ln(1+x) - x + \ln(1+x)\Big)\right]_0^1 - n\int_0^1 x^{n-1}\Big(x\ln(1+x) - x + \ln(1+x)\Big)\,dx \\
&= 2\ln 2 - 1 - n\int_0^1 x^n\ln(1+x) + x^{n-1}\ln(1+x) - x^n\,dx \\
&= 2\ln 2 - 1 - n\left(I_n + I_{n-1}\right) + n\left[\frac{x^{n+1}}{n+1}\right]_0^1 \quad ...(*) \\
&= 2\ln 2 - 1 + \frac{n}{n+1} - nI_n - nI_{n-1} \\
(n+1)I_n &= 2\ln 2 + \frac{n-n-1}{n+1} - nI_{n-1} \\
(n+1)I_n &= 2\ln 2 - \frac{1}{n+1} - nI_{n-1}
\end{aligned}$$

iii. (4 marks)

- ✓ [1] for an expression for I_{n-1}
- ✓ [1] for arriving at (15.1)
- ✓ [1] for calculating I_0
- ✓ [1] for showing the final result

$$\begin{aligned}
(n+1)I_n &= 2\ln 2 - \frac{1}{n+1} - nI_{n-1} \\
&= 2\ln 2 - \frac{1}{n+1} - \left(2\ln 2 - \frac{1}{n} - (n-1)I_{n-2}\right) \\
&= -\frac{1}{n+1} + \frac{1}{n} + (n-1)I_{n-2} \\
&= -\frac{1}{n+1} + \frac{1}{n} + 2\ln 2 - \frac{1}{n-1} - (n-2)I_{n-3} \\
&= 2\ln 2 - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n-1} - (n-2)I_{n-3}
\end{aligned}$$

Since n is odd,

$$(n+1)I_n = 2\ln 2 - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n-1} + ... + \frac{1}{3} - \frac{1}{2} - I_0 \tag{15.1}$$

$$\begin{aligned}
\because I_0 &= \int_0^1 \ln(1+x)\,dx \\
&= [(x+1)\ln x - x]_0^1 \\
&= 2\ln 2 - 1 \\
\therefore (n+1)I_n &= \cancel{2\ln 2} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n-1} + ... + \frac{1}{3} - \frac{1}{2} - (\cancel{2\ln 2} - 1) \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{n+1}
\end{aligned}$$

(b) i. (1 mark)

$$\begin{aligned}
x &= a\sin(nt) \\
\dot{x} &= an\cos(nt)
\end{aligned}$$

Since $\dfrac{2\pi}{n} = T$, $n = \dfrac{2\pi}{T}$

$$\therefore \dot{x} = \frac{2a\pi}{T}\cos\left(\frac{2\pi}{T}t\right)$$

ii. (2 marks)

✓ [1] for equations (1) and (2)

✓ [1] for showing the final result

When $t = t_1$, $x = D$ and $x = D$, $\dot{x} = V$.

$$\begin{aligned}
D &= a\sin(nt_1) \qquad ...(1) \\
V &= an\cos(nt_1) \qquad ...(2)
\end{aligned}$$

(1) $\div$ (2),

$$\begin{aligned}
\frac{D}{V} &= \frac{1}{n}\tan(nt_1) \\
\therefore t_1 &= \frac{1}{n}\tan^{-1}\left(\frac{nD}{v}\right) \\
&= \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)
\end{aligned}$$

iii. (2 marks)

✓ [1] for getting an expression for $-V$

✓ [1] for getting the final result

Let the particle comes back to P at $t = t_2$. Then,

$$\begin{aligned}
D &= a\sin\left(\frac{2\pi}{T}t_2\right) \qquad ...(1) \\
-V &= \frac{2a\pi}{T}\cos\left(\frac{2\pi}{T}t_2\right) \qquad ...(2)
\end{aligned}$$

(1) $\div$ (2),

$$\begin{aligned} -\frac{D}{V} &= \frac{T}{2\pi}\tan\left(\frac{2\pi}{T}t_2\right) \\ \therefore \tan\left(\frac{2\pi}{T}t_2\right) &= -\frac{2\pi D}{VT} \end{aligned}$$

Question 16 ()

(a) i. (3 marks)

✓ [1] for (1)

✓ [1] for (2)

✓ [1] for showing the final result

Horizontal components:

$$F\cos\theta - 0.2N = 0 \qquad ...(1)$$

Vertical components:

$$N + F\sin\theta - 5g = 0 \qquad ...(2)$$

(1) + 0.2$\times$(2),

$$\begin{aligned} F\cos\theta + 0.2F\sin\theta &= 5g\times 0.2 \\ F(\cos\theta + 0.2\sin\theta) &= 10 \\ F &= \frac{10}{\cos\theta + \frac{1}{5}\sin\theta} \\ &= \frac{50}{5\cos\theta + \sin\theta} \end{aligned}$$

ii. (3 marks)

✓ [1] for arriving at min F when $5\cos\theta + \sin\theta$ is max

✓ [1] for value of R

✓ [1] for final answer

Minimum F when $5\cos\theta + \sin\theta$ is maximum.
Let $5\cos\theta + \sin\theta = R\cos(\theta - \alpha)$.

$$\begin{aligned} 5\cos\theta + \sin\theta &= R\cos\alpha\cos\theta + R\sin\alpha\cos\theta \\ R\cos\alpha &= 5,\ R\sin\alpha = 1 \\ \therefore R &= \sqrt{26} \end{aligned}$$

Now $5\cos\theta + \sin\theta = \sqrt{26}\,(\cos(\theta-\alpha))$
$\Rightarrow$ The maximum value of $5\cos\theta + \sin\theta$ is $\sqrt{26}$

$$\therefore \text{Min } F = \frac{50}{\sqrt{26}}$$

(b) i. (5 marks)

✓ [1] for an expression for v_x

✓ [1] for an expression for x
✓ [1] for an expression for $\ddot{y}$
✓ [1] for an expression for v_y
✓ [1] for an expression for y

Horizontal component:

$$\begin{aligned} m\ddot{x} &= -mkv_x \\ \frac{dv_x}{dt} &= -kv_x \\ \int \frac{1}{v_k}\,dv_x &= \int -k\,dt \\ \ln|v_x| &= -kt + C_1 \end{aligned}$$

$\because$ when $t = 0,\ v_x = u_0$

$$\begin{aligned} C_1 &= \ln u_0 \\ \therefore \ln(v_x) &= -kt + \ln u_0 \\ \ln\left(\frac{v_x}{u_0}\right) &= -kt \\ v_x &= u_0 e^{-kt} \end{aligned}$$

Now

$$\begin{aligned} \frac{dx}{dt} &= u_0 e^{-kt} \\ x &= u_0 \int e^{-kt}\,dt \\ &= -\frac{u_0}{k}e^{-kt} + C_2 \end{aligned}$$

$\because$ when $t = 0,\ x = 0$

$$\begin{aligned} C_2 &= \frac{u_0}{k} \\ \therefore x &= -\frac{u_0}{k}e^{-kt} + \frac{u_0}{k} \\ &= \frac{u_0}{k}\left(1 - e^{-kt}\right) \end{aligned}$$

Vertical component:

$$\begin{aligned} m\ddot{y} &= -mg - mkv_y \\ \frac{dv_y}{dt} &= -g - kv_y \\ \int \frac{1}{g + kv_y}\,dv_y &= -\int dt \\ \frac{1}{k}\ln|g + kv_y| &= -t + C_3 \end{aligned}$$

$\because$ when $t = 0, v_y = v_0$

$$\begin{aligned}
C_3 &= \frac{1}{k}\ln|g + kv_y| \\
\therefore t &= -\frac{1}{k}\ln\left|\frac{g + kv_y}{g + kv_0}\right| \\
\frac{g + kv_y}{g + kv_0} &= e^{-kt} \\
g + kv_y &= (g + kv_0)e^{-kt} \\
v_y &= \frac{g + kv_0}{k}e^{-kt} - \frac{g}{k}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{dy}{dt} &= \left(\frac{g}{k} + v_0\right)e^{-kt} - \frac{g}{k} \\
y &= -\left(\frac{g}{k^2} + \frac{v_0}{k}\right)e^{-kt} - \frac{g}{k}t + C_4
\end{aligned}$$

$\because$ when $t = 0,\ y = 0$

$$\begin{aligned}
C_4 &= \frac{g}{k^2} + \frac{v_0}{k} \\
\therefore y &= -\left(\frac{g}{k^2} + \frac{v_0}{k}\right)e^{-kt} - \frac{g}{k}t + \left(\frac{g}{k^2} + \frac{v_0}{k}\right) \\
&= \left(\frac{g}{k^2} + \frac{v_0}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t
\end{aligned}$$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} \frac{u_0}{k}\left(1 - e^{-kt}\right) \\ \left(\frac{g}{k^2} + \frac{v_0}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \end{pmatrix}$$

ii. (2 marks)

✓ [1] for an expression for t

✓ [1] for the final answer

From $x = \frac{u_0}{k}\left(1 - e^{-kt}\right)$,

$$\begin{aligned}
1 - e^{-kt} &= \frac{kx}{u_0} \\
e^{-kt} &= 1 - \frac{kx}{u_0} \\
t &= -\frac{1}{k}\ln\left(1 - \frac{kx}{u_0}\right)
\end{aligned}$$

Sub this into $y = \left(\frac{g}{k^2} + \frac{v_0}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$

$$\begin{aligned}
y &= \left(\frac{g}{k^2} + \frac{v_0}{k}\right) \times \frac{kx}{u_0} + \frac{g}{k} \times \frac{1}{k}\ln\left(1 - \frac{kx}{u_0}\right) \\
\therefore y &= \frac{x}{u_0}\left(\frac{g}{k} + v_0\right) + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u_0}\right)
\end{aligned}$$